Extremal Problems in Eulerian Digraphs

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Graphs and digraphs (directed graphs) behave quite differently

- any graph G with n vertices and m edges has a subgraph with minimal degree $\geq m/n$, implying
- G has a cycle of length $\geq m/n + 1$,
- however, those facts are trivially false for general digraph, even for dense digraph, i.e:
  - transitive tournament,
  - complete bipartite digraph $A \cup B$ with all edges from A to B.
- Consider restricted family of digraphs

- **Eulerian digraph**:
  - out-degree $d^+(u)$ = in-degree $d^-(u)$ for any vertex u.
  - No loop and parallel edges; may has 2-cycles.
- why do we consider Eulerian digraph?

  a natural candidate; "easy" to evaluate eigenvalues; it has good properties:

  (1) for any cut (A, B), number of edges from A to B = number of edges from B to A:

$$e(A) + e(A, B) = \sum_{u \in A} d^+(u) = \sum_{u \in A} d^-(u) = e(A) + e(B, A)$$

  (2) the edge set can be decomposed into edge-disjoint cycles,

Some interesting problems

- Conjecture(Bollobás and Scott, Dean): for Eulerian digraphs, there exists a cycle decomposition with $O(n)$ cycles ?
  - Even not known for graph!
- Conjecture (Erdős, Goodman and Pósa, 1966): for even graphs, there exists a cycle decomposition with $O(n)$ cycles ?
  - easy to get $O(n \log n)$:

  greedily pick and remove a cycle of length $\geq m/n$ in the current graph; keep doing (say t steps) until

$$m(1-\frac{1}{n})^t = n \quad \Rightarrow \quad t \leq n \log m/n \leq n \log n.$$

- Conjecture(Bollobás and Scott, 1996): Any Eulerain digraph with m edges and n verices has a cycle of length $\Omega(m/n)$.
  - this motivates our work

more natural parameters

- **girth** $g(G)$= the length of the shortest cycle in G
- **minimal feedback arc set**= the smallest set of edges whose removal results in an acyclic digraph; denote its cardinality as $\beta(G)$.
  - important in combinatorial optimization and with many applications to other fields.
  - computing $\beta(G)$ is NP-hard even for tournaments.
- We are also interested to find sub-digraph whose minimal degree is large.
- For graphs, general bounds about those parameters are known.
  - DFS $\Rightarrow$ long cycle of length $\Omega(m/n)$
  - Moore bound $\Rightarrow g(G) = O(\log n)$ (if $m \geq 3n$)
  - $\beta(G) = m - (n-1)$ if G is connected

Our results

Every Eulerian digraph G with n vertices and m edges has

- **Thm 1**: $\beta(G) \geq \frac{m^2}{2n^2} + \frac{m}{2n}$. Optimal for any m, n with $m = kn$.
  $\Downarrow$
- **Thm 2**: girth $g(G) \leq 6n^2/m$. Tight up to a constant.
  $\Downarrow$
- **Thm 3**: an Eulerian sub-digraph with minimal degree $\geq m^2/24n^3$. Tight up to a constant.
  $\Downarrow$
- **Thm 4**: a cycle of length $\geq \max\{m^2/24n^3, \sqrt{m/n}\}$.
  - When $m = cn^2$, this gives a cycle of length linear in n.
  - When graph is sparse, $\sqrt{m/n}$ is better.

Thm 1: Minimal feedback arc set $\beta(G) \geq \frac{m^2}{2n^2} + \frac{m}{2n}$

- Extremal examples: there exists a Cayley digraph G for any m, n with $m = kn$ such that $\beta(G) = \frac{m^2}{2n^2} + \frac{m}{2n}$.
- A weaker bound what we are going to prove: $\beta(G) \geq \frac{m^2}{8n^2}$.
- consider a linear order of vertices, say $v_1, v_2, ..., v_n$. Arc (v_i, v_j) is **backward** if $i > j$; otherwise **forward**.
- it suffices to prove: for arbitrary order of vertices, the number of backward arcs $\geq \frac{m}{4k}$, where $k = 2n^2/m$.
- A careful analysis will lead to the optimal bound: we need to count the "contribution" of an arc whose two endpoint are far. Complicate.

From $\beta(G) \geq \frac{m^2}{2n^2} + \frac{m}{2n}$ to Thm 2: girth $g(G) \leq 6n^2/m$

- Need a Theorm of Fox-Keevash-Sudakov: If a digraph G satisfies $\beta(G) > 18n^2/r^2$ for $r > 11$, then $g(G) \leq r$.
- If there exists a 2-cycle, then $g(G) = 2 \leq 6n^2/m$.
- Assume G has no 2-cycle, then $6n^2/m \geq 12 > 11$, then

$$\beta(G) > \frac{m^2}{2n^2} = \frac{18n^2}{(6n^2/m)^2} \quad \Rightarrow \quad g(G) \leq 6n^2/m.$$

- Tight up to constant and the optimal constant seems to be 1: again the Cayley digraphs.
  - Caccetta-Häggkvist Conjecture; when $r = n/3$ and Eulerian

Thm3: an Eulerian subgraph with large minimal degree $\geq m^2/24n^3$

- Thm 2 says that Eulerain digraph with at least $m/2$ edges has a cycle of length $\leq 12n^2/m$.
- at least $m/2$ edges can be decomposed into t edge-disjoint cycles of each length $\leq 12n^2/m$, thus $t \geq m^2/24n^2$.
- Define the union of cycles as H, which is an Eulerian digraph.
- If there is a vertex u whose degree $< m^2/24n^3$, then we delete u and all short cycles through u which we choose.
- When stop, we get a sub-digraph with such property.

Extremal digraphs in which the minimal degree of any Eulerian sub-digraph is $O(m^2/n^3)$.

- $V(H(s,t)) = U_1 \cup \ldots \cup U_s \cup V_1 \cup \ldots \cup V_t$ and $|U_i| = |V_j| = s$.

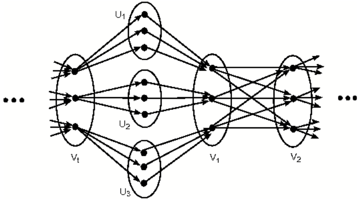

Figure: $H(s,t)$ with $s = 3$

- Any cycle in $H(s,t)$ must pass through some vertex in U_i, whose degree is 1.
- Our extremal digraphs: δ-blowup of $H(s,t)$, which has $n = s(s+t)\delta, m = s^2(t+1)\delta^2$.
- Taking $t = 2s$, then $m^2/n^3 \geq 4\delta/27 \quad \Rightarrow \quad \delta \leq O(m^2/n^3)$.

Thm4: cycle of length $\geq \max\{m^2/24n^3, \sqrt{m/n}\}$

- Lemma: any digraph with minimal out-degree k has a cycle of length $\geq k+1$.
  $\Downarrow$
- a cycle of length $\geq m^2/24n^3$
- An inductive argument shows the existence of cycle of length $\geq \sqrt{m/n}$.
  - By lemma, we assume vertex u has out-degree $t \leq \sqrt{m/n}$, then we delete u and all cycles $C_1, \ldots, C_t$ passing through u. Also assume $|C_i| < \sqrt{m/n}$.
  - The remaining graph has $n-1$ vertices and m' edges:

$$m' \geq m - t\sqrt{m/n} \geq m(1 - \frac{1}{n})$$

  - By induction, there exists a cycle of length at least $\sqrt{m'/(n-1)} \geq \sqrt{m/n}$.

Open problems

- **Bollobás-Scott**: Should have a cycle of length $\Omega(m/n)$.
  - DFS fails
- **Caccetta-Häggkvist**: every digraph with n vertices and minimal out-degree at least r should contain a cycle of length $\leq \lceil n/r \rceil$.
  - Most interesting case: $r = n/3$, which is open even for Eulerian digraphs!
- **Seymour**: For Eulerian digraphs,

$$\sum_{i \in V} d^{++}(i) \geq \sum_{i \in V} d^{+}(i) = m.$$

Happy Birthday! Robin!